Dynamical Analysis of Ratio-dependent Predator-Prey Models with Prey Refuge and Optimal Harvesting on Both Populations

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Abstract— In this article, we discuss a dynamical analysis of ratio-dependent response predator-prey model with prey refuge and harvesting on both populations. Harvesting on predator and prey is used because both populations are assumed to have an economic value. In this model, it is also assumed that prey has an instinct to protect themselves from the threat of predators. The conducted dynamical analysis consists of determination of equilibria, existence conditions and their stability. Analytical results show that there are two equilibria: namely predator extinction point and the coexistence point which are exist and stable under certain condition. Furthermore, Pontryagin maximum principle is used to find optimal harvesting on predator and prey. Some numerical simulations are performed to illustrate the analytical result.

Keywords— dynamical analysis; predator- prey model; ratio-dependent; prey refuge; optimal harvesting.

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I. INTRODUCTION

Lotka and Volterra were introducing differential equations describing the interaction interspecies in an ecosystem i.e. predator and prey that called with Lotka-volterra models [1]. In the progression, some modifications were done on Lotka-Volterra model. Ilmiyah et al. [2] and Kar [3] modified Lotka-Volterra model by adding harvesting on both populations and prey refuge, Kar in [3] modified the model as follows

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \frac{s(1-p)xy}{c+(1-p)x} - q_1 u_1 x,$$

$$\frac{dy}{dt} = -dy + \frac{f(1-p)xy}{c+(1-p)x} - q_2 u_2 y, \quad (1)$$

where $x = x(t)$ and $y = y(t)$ represent the number of prey and predator population at time t respectively. In this model, Kar [3] used Holling type II response that the growth rate of prey only depend on the prey population density. The fact that the response function prey should depend on the density of predator and prey [4], next Christoper [5] examined Lotka-Volterra model by using ratio-dependent response function and harvesting only on the predator side. Some modified were done on rensponse function, Edwin [6] used Holling type II and ratio-dependent response on his work.

In this article we modified Lotka-Volterra predator prey models [7] by adding prey refuge and harvesting on both populations [3]. Because of an economic value, we used the model with harvesting on predator and prey. In this model, it is also assumed that prey has an instinct to protect themselves from the threat of predators, because in the wild, prey instinct to protect themselves is a factor to be taken into account in modelling [8]. In this article, optimal harvesting is discussed using theory of control optimal as in [9]. Ratio-dependent predator and prey model with protection on prey and harvesting on both populations can be stated into the following equation.

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \frac{c_1(1-p)xy}{(1-p)x+my} - q_1 u_1 x,$$

$$\frac{dy}{dt} = y\left(-d + \frac{c_2(1-p)x}{(1-p)x+my}\right) - q_2 u_2 y, \quad (2)$$

where $x(t)$ and $y(t)$ denote the number of prey and predator population at time t, respectively. r denotes the prey intrinsic growth rate, c_1 is capturing rate, c_2 represents conversion rate of the predator, d is the death rate of predator,, m denotes half capturing saturation constant, and K is the carrying capacity in the absence of predation representing the maximum number of individuals that can be supported by the environtment [10]. All parameters are positive constant.

II. RESEARCH METHOD

1. Literature Study

At this stage, the identification of problems in the research will be carried out. The study of library sources can support the resolution of the problems to be carried out. Learning from these library sources will be useful in understanding the model of the Predator-prey so that it can support the analyzes carried out

2. Tools and Materials

In this paper, the tools and materials used are as follows: (i) MATLAB 2017. (ii) Computer with processor specifications AMD Ryzen 5, RAM 8GB and Windows 10 64-bit operation.

3. Research Flow

In this paper, the following steps were carried out:

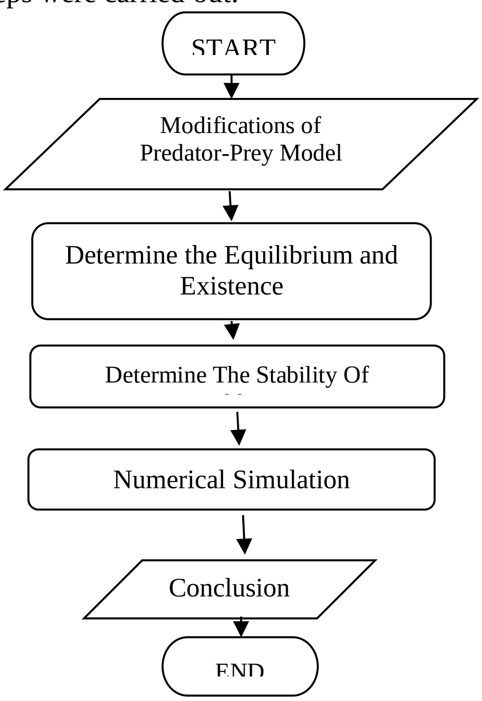

Figure 1. Flowchart

The explanation of the research steps are as follows:

1) Model Construction

At this stage, This model uses the Lotka Volterra Models, modification were done by adding the optimal harvesting on both populations and prey refuge [3] and by using ratio dependent respond function [5].

2) Determine the. Equilibrium Poins

Based on the model that has been formed, then look for the fixed point of the Predator-prey model. Analytical results show that there two equilibria which exist under a certain condition.

3) Determining the Stability of the Equilibrium Points

At this stage, stability analysis will be carried out on the model of the Predator-prey. The model used is a non-linear model, so before carrying out the analysis process, the equation will be linearized first. The search for eigenvalues that fit the model so that stability analysis can be carried out on the linearized model. As for equations that are already linear, there is no need for linearization.

4. Numerical Simulation

This stage is done to make it easier to make observations about the dynamics of the system. This simulation will be performed numerically on the model. Simulation at this stage several conclusions are drawn based on the simulation results obtained.

III. RESULT AND DISCUSSION

1. EQUILIBRIUM AND EXISTENCE

The equilibria points system (2) are solution from the following system

$$\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right) - \frac{c_1(1-p)xy}{(1-p)x+my} - q_1u_1x = 0,$$
$$\frac{dy}{dt} = y\left(-d+\frac{c_2(1-p)x}{(1-p)x+my}\right) - q_2u_2y = 0.$$

System (2) has two equilibria points i.e $E_1(\frac{(r-q_1u_1)K}{r},0)$ and $E_2(x^*,y^*)$. The equilibria E_1 is predator extinction point, whereas, E_2, is coexistence point where both populations could live side by side. E_1 exist, if $q_1u_1 < r$. The coexistence point is $E_2(x^*,y^*)$ with $x^* = \frac{(d+q_2u_2)my^*}{(c_2-(d+q_2u_2))(1-p)}$ and $y^* = \frac{-B\pm\sqrt{B^2-4AC}}{-2A}$, where

$$A = -rc_2(d+q_2u_2)m^2 \quad (3.a)$$
$$B = rc_2^2Km - rc_2Km(d+q_2u_2) - rc_2^2pKm + rc_2pKm(d+q_2u_2) - c_1c_2^2pK$$
$$+2(c_1c_2dpK + c_1c_2q_2u_2pK - c_1q_2u_2dpK) - c_1d^2pK - c_1(q_2u_2)^2pK - c_1c_2^2p^2K$$
$$+2(c_1c_2dp^2K + c_1c_2q_2u_2p^2K - c_1q_2u_2dp^2K) - c_1d^2p^2K - c_1(q_2u_2)^2p^2K \quad (3.b)$$
$$C = q_1u_1c_2K - q_1u_1dK - q_1u_1q_2u_2K - q_1u_1c_2pK + q_1u_1dpK + q_1u_1q_2u_2pK \quad (3.c)$$

The equilibria E_2 exist if the following conditions are satisfied.

i. $c_2 > d + q_2u_2$

ii. If $B > 0, C < 0$ and $B^2 - 4AC > 0$, then E_2 has one positive value $y_1^* = \frac{-B-\sqrt{B^2-4AC}}{-2A}$.

iii. If $B > 0$ and $C > 0$, then E_2 has one positive value $y_2^* = \frac{-B-\sqrt{B^2-4AC}}{-2A}$.

If $B < 0$ and $C > 0$, then E_2 has one positive value $y_3^* = \frac{-B-\sqrt{B^2-4AC}}{-2A}$.

2. STABILITY OF EQUILIBRIUM

The local stability of system (2) for each equilibria points are shown in Theorem 3.1.
Theorem 3.1

i. $E_1 = \left(\frac{(r-q_1u_1)K}{r},0\right)$ is stable if $2q_1u_1 < r$.

ii. $E_2 = (x^*,y^*)$ is stable if $r < \frac{2rx}{K} + \frac{c_1(1-p)my^2}{((1-p)x+my)^2} + q_1u_1$

(4)

Proof

Stability analysis is used to investigate the characteristic of equilibria points of system (2). System (2) are autonomous nonlinear system, then to determine the stability of equilibria points we linearize system (2). Linearized results of system (2) using Jacobian matrix is as follows

$$J = \begin{bmatrix} r - \frac{2rx}{K} - \frac{c_1(1-p)my^2}{\left((1-p)x+my\right)^2} - q_1u_1 & -\frac{c_1\left((1-p)x^2\right)}{\left((1-p)x+my\right)^2} \\ \frac{c_2(1-p)my^2}{\left((1-p)x+my\right)^2} & -d + \frac{c_2(1-p)^2x^2}{\left((1-p)x+my\right)^2} - q_2u_2 \end{bmatrix}.$$

i. Jacobian matrix system (2) for E_1 is $J(E_1) = \begin{bmatrix} -r + 2(q_1u_1) & 0 \\ 0 & -d - q_2u_2 \end{bmatrix}$. Eigenvalue of Jacobian matrix $J(E_1)$ is $\lambda_1 = -r + 2(q_1u_1)$ and $\lambda_2 = -d - q_2u_2$. Then we have E_1 is asymptotically stable if $2q_1u_1 < r$.

ii. Jacobian matrix for E_2 is $J(E_2) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$ with $J_{11} = r - \frac{2rx}{K} - \frac{c_1(1-p)my^2}{\left((1-p)x^*+my^*\right)^2} - q_1u_1$,

$J_{12} = -\frac{c_1\left((1-p)x\right)^2}{\left((1-p)x+my\right)^2}$, $J_{21} = \frac{c_2(1-p)my^2}{\left((1-p)x+my\right)^2}$ and $J_{22} = -d + \frac{c_2(1-p)^2x^2}{\left((1-p)x+my\right)^2} - q_2u_2$. The eigenvalue of J

acobian matrix $J(E_2)$ are difficult to determine, then to determine the criteria of the stability can be investigated by utilizing trace $J(E_2)$ and determinant $J(E_2)$. According Pavilov [12] the equilibrium point system (2) has stable characteristic if determinant $> J(E_2)\ 0$ and trace $J(E_2) < 0$. Thereby, it need to be investigated the value from Jacobian matrix (E_2). Since all the parameters positive and $p < 1$, so J_{12} being negative, J_{21} has positive value, for J_{11} and J_{22} needs to be proven in order to get negative value. J_{11} have negative value if $r < L$, with $L = \frac{2rx}{K} + \frac{c_1(1-p)my^2}{\left((1-p)x+my\right)^2} + q_1u_1$. Now, from system (2) at equilibrium point,

we have

$-d + \frac{c_2(1-p)x}{(1-p)x+my} - q_2u_2 = 0$ it can be written as

$$-d - q_2u_2 = -\frac{c_2(1-p)x}{(1-p)x+my} \tag{5}$$

Further, by substituting equation (5) to J_{22}, we obtain

$$J_{22} = -d + \frac{c_2(1-p)^2x^2}{\left((1-p)+my\right)^2} - q_2u_2$$
$$= -\frac{c_2(1-p)x}{(1-p)x+my} + \frac{c_2(1-p)^2x^2}{\left((1-p)+my\right)^2}$$
$$= \frac{c_2(1-p)x}{(1-p)x+my}\left(-1 + \frac{c_2(1-p)x}{(1-p)x+my}\right)$$

Because $p < 1$, while $x^* > 0$ and $y^* > 0$, it is clear that $(1-p)x^* + my^* > (1-p)x^*$, as a result $-1 + \frac{(1-p)x^*}{(1-p)x^*+my^*}$ is negative, therefore $J_{22} < 0$. By knowing the signs of J_{11}, J_{12}, J_{21} and J_{22} it can be seen that determinant $J(E_2) > 0$ and trace $J(E_2) < 0$. So that E_2 is asymptotically stable if $r < L$.

3. BIONOMIC EQUILIBRIUM

The bionomic equilibrium point is $(x_\infty, y_\infty, u_{1\infty}, u_{2\infty})$ which have positive solution from the following system.

$$0 = r\left(1 - \frac{x}{K}\right) - \frac{c_1(1-p)y}{(1-p)x+my} - q_1u_1 \tag{6.a}$$

$$0 = -d + \frac{c_2(1-p)x}{(1-p)x+my} - q_2u_2 \tag{6.b}$$

$$\pi = (p_1 q_1 x - n_1)u_1 + (p_2 q_2 y - n_2)u_2 = 0 \tag{6.c}$$

where $\pi = (p_1 q_1 x - n_1)u_1 + (p_2 q_2 y - n_2)u_2$ represents the revenues for the prey and predator species at time t, with n_1 is cost per unit effort for prey species, n_2 is cost per unit effort for predator species, p_1 denotes price per unit biomass of the prey, p_2 denotes price per unit biomass of the predator. In order to determine the equilibria points, we will consider the following cases as in [3]:

i. Case 1. If $n_2 > p_2 q_2 y$ i.e the predator harvest will be stopped if the revenue is lower than the cost for predator.

ii. Case 2. If $n_1 > p_1 q_1 x$ i.e the prey harvest will be stopped if he revenue is lower than the cost for prey.

iii. Case 3. If $n_1 > p_1 q_1 x$, $n_2 > p_2 q_2 y$ i.e the revenue is lower than cost for both of populations, then the harvesting of prey and predator will be stopped.

iv. Case 4. If $n_1 < p_1 q_1 x$ and $n_2 < p_2 q_2 y$ ie when income is positive and both species have the advantage then harvesting in both populations will be done. In this case, $x_\infty = n_1/p_1 q_1$ and $y_\infty = n_2/p_2 q_2$, then substituting x_∞ and y_∞ into (6.a) and (6.b), we get

$$u_{1\infty} = \frac{r}{q_1}\left(1 - \frac{n_1}{p_1 q_1 K}\right) - \frac{c_1(1-p)n_2 p_1 q_1}{(1-p)n_1 p_2 q_2 + m n_2 p_1 q_1}$$

$$u_{1\infty} > 0 \text{ if } \frac{r}{q_1}\left(1 - \frac{n_1}{p_1 q_1 K}\right) > \frac{c_1(1-p)n_2 p_1 q_1}{(1-p)n_1 p_2 q_2 + m n_2 p_1 q_1} \tag{7}$$

and

$$u_{2\infty} = -\frac{d}{q_2} + \frac{c_2(1-p)n_1 p_2 q_2}{(1-p)n_1 p_2 q_2 + m n_2 p_1 q_1}$$

$$u_{2\infty} > 0 \text{ if } \frac{c_2(1-p)n_1 p_2 q_2}{(1-p)n_1 p_2 q_2 + m n_2 p_1 q_1} > \frac{d}{q_2} \tag{8}$$

Thus, the bionomic equilibrium point $[x_\infty, y_\infty, u_{1\infty}, u_{2\infty}]$ exists if and only if conditions (7) and (8) are fulfilled.

4. OPTIMAL HARVESTING

The fundamental problem in commercial exploitation of renewable resources is to determine the optimal target between current and future harvest. In order to get optimal harvesting that maximize the revenue and minimize cost for both the species, need to be established objective functional, we adopt the objective functional as in [3]

$$J(u_1(t), u_2(t)) = \int_0^\infty e^{-\delta t}\{(p_1 q_1 x - n_1)u_1(t) + (p_2 q_2 y - n_2)u_2(t)\} \tag{9}$$

with δ denotes annual rate of discount. To maximize (9) we use Pontryagin's maximal principle. Based on the Pontryagin's maximal principle, we define Hamiltonian function as follows.

$$H = e^{-\delta t}\{(p_1 q_1 x - n_1)u_1 + (p_2 q_2 y - n_2)u_2\} + \sigma_1 g_1 + \sigma_2 g_2$$

where,

$\sigma_1(t)$ and $\sigma_2(t)$ are adjoint variable and,

$$g_1 = rx\left(1 - \frac{x}{K}\right) - \frac{c_1(1-p)xy}{(1-p)x + my} - q_1 u_1 x,$$

$$g_2 = -dy + \frac{c_2(1-p)xy}{(1-p)x + my} - q_2 u_2 y.$$

Then, we determine optimal conditions using Hamiltonian function as follow.

$$\frac{\partial H}{\partial u_i} = 0, \qquad i = 1,2.$$

$$\frac{\partial H}{\partial u_1} = 0 \Longrightarrow \sigma_1 = e^{-\delta t}\left(p_1 - \frac{n_1}{q_1 x}\right), \qquad (10)$$

$$\frac{\partial H}{\partial u_2} = 0 \Longrightarrow \sigma_2 = e^{-\delta t}\left(p_2 - \frac{n_2}{q_2 y}\right). \qquad (11)$$

Adjoint equation is obtained by differentiate the Hamiltonian function about each state variable, given by

$$\dot{\sigma_1} = -\frac{\partial H}{\partial x},$$

$$= -[e^{\delta t}(p_1 q_1 u_1) + \sigma_1\left\{\left(1 - \frac{2x}{K}\right) - \frac{c_1(1-p)my^2}{\left((1-p)x+my\right)^2} - q_1 u_1\right\} + \sigma_2\left\{\frac{c_2(1-p)my^2}{\left((1-p)x+my\right)^2}\right\}, \qquad (12)$$

$$\dot{\sigma_2} = -\frac{\partial H}{\partial y}$$

$$= -[e^{\delta t}(p_2 q_2 u_2) + \sigma_1\left\{-\frac{c_1(1-p)^2x^2}{\left((1-p)x+my\right)^2}\right\} + \sigma_2\left\{-d + \frac{c_2(1-p)mx^2}{\left((1-p)x+my\right)^2} - q_2 u_2\right\}. \qquad (13)$$

As a results from equation (10) – (13), we obtain

$$\dot{\sigma_1} = -\frac{\partial H}{\partial x}$$

$$-\delta e^{-\delta t}\left(p_1 - \frac{n_1}{q_1 x}\right) = -[e^{\delta t}(p_1 q_1 u_1) + \sigma_1\left\{\left(1 - \frac{2x}{K}\right) - \frac{c_1(1-p)my^2}{\left((1-p)x+my\right)^2} - q_1 u_1\right\}$$

$$+\sigma_2\left\{\frac{c_2(1-p)my^2}{\left((1-p)x+my\right)^2}\right\}, \quad (14)$$

$$\dot{\sigma_2} = -\frac{\partial H}{\partial y}$$

$$-\delta e^{-\delta t}\left(p_2 - \frac{n_2}{q_2 y}\right) = -[e^{\delta t}(p_2 q_2 u_2) + \sigma_1\left\{-\frac{c_1(1-p)^2x^2}{\left((1-p)x+my\right)^2}\right\}$$

$$+\sigma_2\left\{-d + \frac{c_2(1-p)my^2}{\left((1-p)x+my\right)^2} - q_2 u_2\right\}. \qquad (15)$$

Furthermore, substituting σ_1 and σ_2 to (14) and (15) to get $x^*, y^*,$ and then we can determine u_1^* and u_2^*, that is $u_1^* = \frac{1}{q_1}\left\{r\left(1 - \frac{x^*}{K}\right) - \frac{c_1(1-p)x^*}{(1-p)x^*+my^*}\right\}$ and $u_2^* = \frac{1}{q_2}\left\{-d + \frac{c_2(1-p)x^*}{(1-p)x^*+my^*}\right\}$.

By using parameter in Table 1, we get bionomic equilibrium is $x^* = 0.9444$ and $y^* = 7{,}284$ and optimal harvesting effort for both species are $u_1^* = 0.8954$ and $u_2^* = 0.7386$. The bionomic equilibrium and optimal harvesting effort meet the conditions as has been described previously, that is

1. Case 4 on bionomic equilibria. From Table 1, it is found that
   $n_1 = 0.2 < p_1 q_1 x = 0.472$ and $n_2 = 0.4 < p_2 q_2 y = 0.5098$
2. Equation (7), that is
   $\frac{r}{q_1}\left(1 - \frac{n_1}{p_1 q_1 K}\right) = 1.92 > \frac{c_1(1-p)n_2 p_1 q_1}{(1-p)n_1 p_2 q_2 + mn_2 p_1 q_1} = 1.2532,$
3. Equation (8), that is
   $$\frac{c_2(1-p)n_1 p_2 q_2}{(1-p)n_1 p_2 q_2 + mn_2 p_1 q_1} = 0.6433 > \frac{d}{q_2} = 0.2857.$$

Table 1. Parameters of optimal harvesting effort

Parameters													
r	K	c_1	c_2	p	m	d	q_1	q_2	n_1	n_2	δ	p_1	p_2
1	10	1.2	0.8	0.2	0.3	0.2	0.5	0.7	0.2	0.4	1.2	1	1.1

5. NUMERICAL SOLUTIONS

In this section, we will do a simulation for system (2). The purpose is to know the behavior of the interactions between predators and prey. Several factors that influence it, that is their instincts take refuge in prey and their harvesting in both populations are investigated. Numerical simulations are conducted using pplane in MATLAB software. The aim of this simulation is to illustrate the stability of the equilibria points $E_1$and E_2.

Simulation 1 illustrates the stability of the predator extinction point (E_1). The parameter values used in the simulation E_1 is $r = 3.75, m = 0.5, K = 1, c_1 = 0.5, c_2 = 1, p = 0.4, u_1 = 0.9, q_1 = 1.5, q_2 = 1, u_2 = 1$ and $d = 0.5$. Based on these parameters value, we have equilibrium point $E_1(0{,}6\ ;0)$ with $2q_1u_1 = 3 < r = 3.75$ These condition qualifies stability E_1, so that E_1 point are asymptotically stable. The simulation results are presented in Figure 2.

Figure 2 shows that using some initial values indicate orbit system (2) always convergent to $E_1 = (0.6,0)$. This is in accorandce with the analysis result that indicates $E_1(0.6,0)$ is asymptotically stable. $E_1 = (0.6,0)$ point also called predator extinction point, this occurs when the rate of predation prey is relatively low compared to the rate of harvesting predators, consequently the harvesting level predator becomes greater than the rate of growth.

Simulation 2 illustrates the stability of the coexistence point (E_2).The parameter values used in the simulation E_2 is $r = 3.75, m = 0.5, K = 1, c_1 = 2, c_2 = 3, p = 0.4, u_1 = 1, q_1 = 1.5,$ $q_2 = 1, u_2 = 1, d = 0.5$. Based on these parameter values, it is obtained $B > 0, C < 0$ and $D > 0$, so that based on existence condition of equilibria we have $E_1 = (0.6,0)$ and $E_2 = (0.28,0.33)$. $E_1 = (0.6,0)$ is not stable, whereas equilibrium point E_2 are local asymptotically stable since qualify the local stability, that is $r = 3.75 < L = 4.14$, and the numerical simulation is given as in Figure 3.

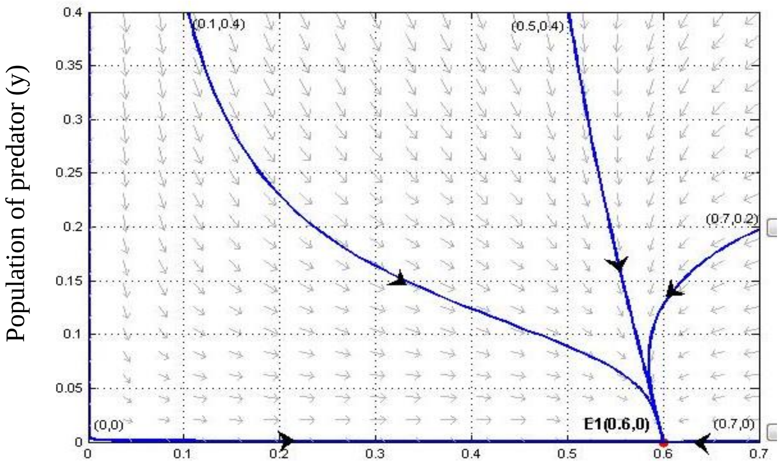

Figure 2. Phase portrait system (2) for simulation 1

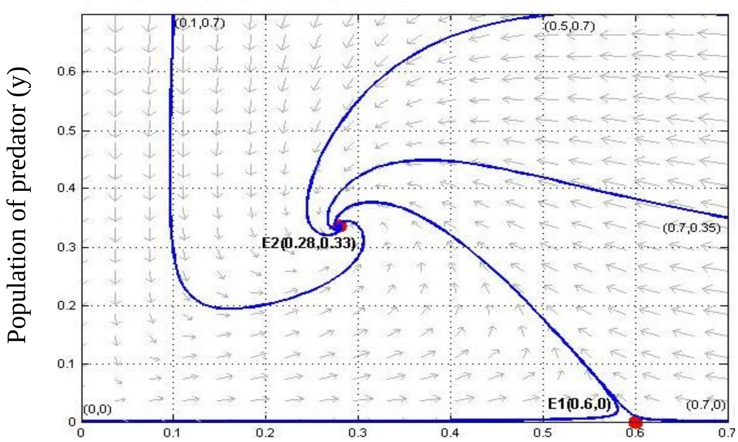

Figure 3. Phase portrait system (2) for simulation 2

Figure 3 shows that by using several initial values the orbit solution of system (2) always convergent to the point $E_2 = (0.28,0.33)$. This is in accorandce with the analysis results which indicates that the equilibrium point $E_2 = (0.28,0.33)$ is asymptotically stable whereas equilibrium point $E_1 = (0.6,0)$ is not stable.

IV. CONCLUSION

In this paper, we have analyzed the stability and optimal harvesting of ratio-dependent predator-prey models with prey refuge and harvesting on both populations. Based on analysis results we get two equilibria point, namely, equilibrium predator extinction point (E_1) and coexsistence equilibrium point (E_2). E_1 and E_2 are exist and asymptotically stable under certain conditions. Optimal harvesting obtained by analizing

bionomic equilibria, then Pontryagin's maximum principle is applied on the model predator-prey. Simulation results show that the analysis results is in accordance with the numerical result.

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